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Partial Padé Approximation

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1. MOTIVATIONS

Let f be a formal power series. Padé approximants are rational functions whose expansion in ascending powers of the variable coincides with f as far as possible, that is, up to the sum of the degrees of the numerator and denominator. The numerator and the denominator of a Padé approximant are completely determined by this condition and no freedom is left.

If some poles of f are known, it can be interesting to use this information. Padé-type approximants are rational functions with an arbitrary denominator, whose numerator is determined by the condition that the expansion of the Padé-type approximant matches the series f as far as possible, that is, up to the degree of the numerator [1]. It is also possible to choose some of the zeros of the denominator of the Padé-type approximants (instead of all) and then determine the others and the numerator in order to match the series f as far as possible. Such approximants, intermediate between Padé and Padé-type approximants, have been called higher order Padé-type approximants. Padé approximants are a particular case of Padé-type approximants. In some cases Padé-type approximants provide a better approximation of f than the classical Padé approximants because the knowledge of the poles has been incorporated into their construction. Now if some poles and some zeros of f are known one can try to use this information and then determine the remaining poles and zeros of the rational approximation in order to match the series f as far as possible. Such approximants will be called partial Padé approximants and their study is the purpose of this work. Particular cases of approximants to the exponential function with prescribed denominators are studied in [3].

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2. DEFINITION

Let

$$f(t)=\sum_{i=0}^{\infty} c_i t^i, \qquad c_i \in \mathbb{C}.$$

Let v_k and w_r be given polynomials of degrees k and r, respectively. Let p_m and q_n be polynomials (to be determined) of degrees m and n, respectively. We set

$$\tilde{v}_k(t)=t^k v_k(t^{-1})$$
$$\tilde{w}_r(t)=t^r w_r(t^{-1})$$
$$\tilde{p}_m(t)=t^m p_m(t^{-1})$$
$$\tilde{q}_n(t)=t^n q(t^{-1})$$
$$\tilde{P}(t)=\tilde{p}_m(t)\,\tilde{v}_k(t)$$
$$\tilde{Q}(t)=\tilde{q}_n(t)\,\tilde{w}_r(t).$$

The polynomials p_m and q_n will be determined such that

$$\tilde{P}(t)-\tilde{Q}(t)f(t)=O(t^{m+n+1}).$$

The rational function $\tilde{P}(t)/\tilde{Q}(t)$ will be called a partial Padé approximant of f and it will be denoted by

$$\{m, v_k/n, w_r\}_f(t).$$

If $v_k(t)=t^k$ and $w_r(t)=t^r$, or if $k=r=0$, then

$$\{m, v_k/n, w_r\}_f \equiv [m/n]_f.$$

If $v_k(t)=t^k$ (or if $k=0$) and if $n=0$ then

$$\{m, v_k/n, w_r\}_f \equiv (m/r)_f.$$

If $v_k(t)=t^k$ (or if $k=0$) and if $n>0$ then $\{m, v_k/n, w_r\}_f$ is the Padé-type approximant $(m/n+r)$ of the higher order $m+n$ since its expansion coincides with that of f up to and including degree $m+n$.

Thus partial Padé approximants generalize Padé and Padé-type approximants.

3. CONSTRUCTION

Let us now study how these partial Padé approximants can be constructed or, in other words, how the polynomials p_m and q_n can be obtained. We have

$$\tilde{p}_m(t)\,\tilde{v}_k(t)-\tilde{q}_n(t)\,\tilde{w}_r(t)f(t)=O(t^{m+n+1})$$

or

$$\tilde{p}_m(t)-\tilde{q}_n(t)\,\tilde{w}_r(t)f(t)/\tilde{v}_k(t)=O(t^{m+n+1})$$

since v_k has the exact degree k and thus $\tilde{v}_k(0)\neq 0$.

Thus the construction is based on one of the following remarks

R_1: $$\tilde{p}_m(t)/\tilde{q}_n(t)=[m/n]_{f\tilde{w}_r/\tilde{v}_k}(t)$$

or

R_2: $$\tilde{p}_m(t)/\tilde{q}_n(t)\,\tilde{w}_r(t)=(m/n+r)_{f/\tilde{v}_k}(t)$$

(approximant of the higher order $m+n$).

In both cases the first step is to compute the coefficients of the series $f/\tilde{v}_k$. We set

$$\tilde{v}_k(t)=v_0+\cdots+v_k\,t^k$$
$$g(t)=f(t)/\tilde{v}_k(t)=g_0+g_1t+\cdots.$$

Thus

$$c_0+c_1t+\cdots=(v_0+\cdots+v_kt^k)(g_0+g_1t+\cdots)$$
$$c_0=v_0g_0$$
$$c_1=v_0\,g_1+v_1g_0$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$c_k=v_0g_k+v_1g_{k-1}+\cdots+v_kg_0$$
$$c_{k+1}=v_0g_{k+1}+v_1g_k+\cdots+v_kg_1$$
$$c_{k+2}=v_0\,g_{k+2}+v_1\,g_{k+1}+\cdots+v_kg_2$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

Since $\tilde{v}_k(0)\neq 0$, $v_0\neq 0$ and these relations directly give the g_i's.

Let us first use remark R_2. We define the linear functionals $g^{(j)}$ on the space of complex polynomials by

$$g^{(j)}(x^i)=g_{i+j},\qquad i\geqslant 0$$

with the convention that $g_{i+j}=0$ if $i+j<0$.

From the theory of Padé-type approximants of higher order [1] we know that q_n must satisfy the conditions

$$g^{(m-n-r+1)}(x^i q_n(x)\, w_r(x)) = 0, \qquad i = 0, \dots, n-1.$$

We have

$$(m/n+r)_{f/\tilde{v}_k}(t) = \sum_{i=0}^{m-n-r} g_i t^i + t^{m-n-r+1}\, \tilde{u}(t)/\tilde{q}_n(t)\, \tilde{w}_r(t)$$

with the convention that the sum is identically zero if $m-n-r<0$ and with

$$\tilde{u}(t) = t^{n+r-1}\, u(t^{-1}) \quad \text{and} \quad u(t) = g^{(m-n-r+1)}\left(\frac{q_n(x)\, w_r(x) - q_n(t)\, w_r(t)}{x-t}\right).$$

Thus

$$\tilde{p}_m(t) = \tilde{q}_n(t)\, \tilde{w}_r(t) \sum_{i=0}^{m-n-r} g_i\, t^i + t^{m-n-r+1}\, \tilde{u}(t)$$

or

$$p_m(t) = q_n(t)\, w_r(t) \sum_{i=0}^{m-n-r} g_i\, t^{m-n-r-i} + u(t).$$

Let us now use remark R_1. We first have to compute the coefficients of the series $f\tilde{w}_r/\tilde{v}_k = g\tilde{w}_r$. We set

$$\tilde{w}_r(t) = w_0 + \cdots + w_r t^r$$
$$h(t) = g(t)\, \tilde{w}_r(t) = h_0 + h_1 t + \cdots.$$

Thus

$$h_0 + h_1 t + \cdots = (w_0 + \cdots + w_r\, t^r)(g_0 + g_1 t + \cdots)$$
$$h_0 = w_0 g_0$$
$$h_1 = w_0\, g_1 + w_1 g_0$$
$$\cdots\cdots\cdots\cdots$$
$$h_r = w_0 g_r + w_1 g_{r-1} + \cdots + w_r g_0$$
$$h_{r+1} = w_0\, g_{r+1} + w_1 g_r + \cdots + w_r g_1$$
$$h_{r+2} = w_0\, g_{r+2} + w_1 g_{r+1} + \cdots + w_r g_2$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$

which gives directly the h_i's.

We define the linear functionals $h^{(j)}$ on the space of complex polynomials by

$$h^{(j)}(x^i) = h_{i+j}, \qquad i \geqslant 0$$

with the convention that $h_{i+j} = 0$ if $i + j < 0$. Since

$$\tilde{p}_m(t)/\tilde{q}_n(t) = [m/n]_h(t)$$

we know that q_n must satisfy the conditions

$$h^{(m-n+1)}(x^i q_n(x)) = 0, \qquad i = 0, ..., n-1.$$

We have

$$[m/n]_h(t) = \sum_{i=0}^{m-n} h_i t^i + t^{m-n+1}\, \tilde{v}(t)/\tilde{q}_n(t)$$

with the convention that the sum is identically zero if $m - n < 0$ and with

$$\tilde{v}(t) = t^{n-1} v(t^{-1}) \qquad \text{and} \qquad v(t) = h^{(m-n+1)}\left(\frac{q_n(x) - q_n(t)}{x - t}\right).$$

Thus

$$\tilde{p}_m(t) = \tilde{q}_n(t) \sum_{i=0}^{m-n} h_i t^i + t^{m-n+1}\, \tilde{v}(t)$$

or

$$p_m(t) = q_n(t) \sum_{i=0}^{m-n} h_i t^{m-n-i} + v(t).$$

From the theory of Padé approximation we know that a unique q_n exists if the Hankel determinant

$$H_n^{(0)} = \begin{bmatrix} h_0 & h_1 & \cdots & h_{n-1} \\ h_1 & h_2 & \cdots & h_n \\ \cdots & \cdots & \cdots & \cdots \\ h_{n-1} & h_n & \cdots & h_{2n-2} \end{bmatrix}$$

is different from zero. We shall assume in the sequel that this condition is satisfied for all n. It ensures the existence and unicity of $\{m, v_k/n, w_r\}$.

Both approaches must, of course, be equivalent. Indeed, since

$$h^{(r+s)}(x^i) = g^{(s)}(x^i w_r(x)),$$

both sets of relations defining q_n are the same. Moreover

$$u(t)=g^{(m-n-r+1)}\left(\frac{q_n(x)\,w_r(x)-q_n(t)\,w_r(x)+q_n(t)\,w_r(x)-q_n(t)\,w_r(t)}{x-t}\right)$$

$$=g^{(m-n-r+1)}\left(w_r(x)\,\frac{q_n(x)-q_n(t)}{x-t}\right)+q_n(t)\,g^{(m-n-r+1)}\left(\frac{w_r(x)-w_r(t)}{x-t}\right)$$

and finally, using the preceding relation between the functionals $g^{(m-n-r+1)}$ and $h^{(m-n+1)}$ we get

$$u(t)=v(t)+q_n(t)\,g^{(m-n-r+1)}\left(\frac{w_r(x)-w_r(t)}{x-t}\right).$$

Thus, since we have

$$p_m(t)=q_n(t)\,w_r(t)\sum_{i=0}^{m-n-r} g_i\,t^{m-n-r-i}+u(t)$$

$$=q_n(t)\sum_{i=0}^{m-n} h_i\,t^{m-n-i}+v(t),$$

it remains to prove that

$$\sum_{i=0}^{m-n} h_i t^{m-n-i}=w_r(t)\sum_{i=0}^{m-n-r} g_i\,t^{m-n-r-i}+g^{(m-n-r+1)}\left(\frac{w_r(x)-w_r(t)}{x-r}\right).$$

We have

$$g^{(m-n-r+1)}\left(\frac{w_r(x)-w_r(t)}{x-t}\right)=w_0(g_{m-n}+g_{m-n-1}t+\cdots+g_{m-n-r+2}\,t^{r-2}$$

$$+g_{m-n-r+1}\,t^{r-1})+\cdots+w_{r-1}g_{m-n-r+1}.$$

Identifying the coefficients of the terms of the same degree we obtain

degree 0 $\quad h_{m-n}=w_r g_{m-n-r}+(w_0 g_{m-n}+\cdots+w_{r-1}\,g_{m-n-r+1})$

degree 1 $\quad h_{m-n-1}=w_r g_{m-n-r-1}+w_{r-1}g_{m-n-r}$

$\qquad +(w_0 g_{m-n-1}+\cdots+w_{r-2}\,g_{m-n-r+1})$

. . . .

degree $m-n$ $\quad h_0=w_0 g_0$.

Thus, since these relations are exactly those defining the h_i's, the property is proved and the constructions based on remark R_1 or R_2 provide the same partial Padé approximant.

From the classical theory of Padé approximants we know that

$$\tilde{p}_m(t) = \begin{bmatrix} \sum_{i=0}^{m-n} h_i t^{n+i} & \cdots\cdots & \sum_{i=0}^{m} h_i t^i \\ h_{m-n+1} & \cdots\cdots & h_{m+1} \\ \cdots & \cdots & \cdots \\ h_m & \cdots\cdots & h_{m+n} \end{bmatrix}$$

$$\tilde{q}_n(t) = \begin{bmatrix} t^n & \cdots\cdots & 1 \\ h_{m-n+1} & \cdots\cdots & h_{m+1} \\ \cdots & \cdots & \cdots \\ h_m & \cdots\cdots & h_{m+n} \end{bmatrix}$$

Thus the construction of $\{m, v_k/n, w_r\}$ needs the knowledge of $c_0, \ldots, c_{m+n}$. Let us now give an example. We consider the series

$$f(t) = tg \quad t/t = 1 + \frac{1}{3}t^2 + \frac{2}{15}t^4 + \frac{17}{315}t^6 + \frac{62}{2835}t^8 + \cdots$$

which has zeros at $\pm\pi$, $\pm 2\pi$, $\pm 3\pi, \ldots$ and poles at $\pm\pi/2$, $\pm 3\pi/2$, $\pm 5\pi/2, \ldots$. We shall take

$$\tilde{w}_1(t) = t - \pi/2 \quad \text{and} \quad \tilde{v}_1(t) = t - \pi.$$

We have

$$g_0 = -1/\pi, \quad g_1 = -1/\pi^2, \quad g_2 = -1/\pi(1/\pi^2 + 1/3)$$

$$h_0 = 1/2, \quad h_1 = -1/2\pi, \quad h_2 = -1/2\pi^2 + 1/6,$$

and we finally obtain

$$\{1, v_1/1, w_1\}_f(t) = \frac{(t-\pi)[3\pi + (\pi^2 - 6)\, t]}{(2t - \pi)[3\pi + (\pi^2 - 3)\, t]}.$$

t	$tg \quad t/t$	$\{1, v_1/1, w_1\}_f(t)$	$[2/1]_f(t) = 1 + t^2/3$
1.5	9.4009466	8.9493644	1.7500000
1	1.5574077	1.5305797	1.3333333
0.5	1.0926049	1.0895934	1.0833333
0.3	1.0311208	1.0304284	1.0300000
0.1	1.0033467	1.0033181	1.0033333
−0.1	1.0033467	1.0033802	1.0033333
−0.5	1.0926049	1.0994597	1.0833333
−1	1.5574077	1.7512451	1.3333333

By construction $\{1, v_1/1, w_1\}$ must give better results near the first zero and pole of f. We have

t	$tg\quad t/t$	$\{1, v_1/1, w_1\}_f(t)$	$[2/1]_f(t)$
1.6	−21.395332	−20.188416	1.8533333
1.7	−4.5274130	−4.2305380	1.9633333
2	−1.0925199	−0.9854227	2.3333333
2.5	−0.2988089	−0.2478919	3.0833333
3.1	−0.0134247	−0.0094250	4.2033333
3.2	0.0182730	0.0012446	4.4133333
4	0.2894553	0.1192313	6.3333333

The second example is

$$f(t) = 1\text{-Log}(1+t) = 1 - t + t^2/2 - t^3/3 + \cdots$$

which has a zero at $t = e - 1$ and a logarithmic branch point $t = -1$. Choosing $\tilde{w}_1(t) = t + 1$ and $\tilde{v}_1(t) = t + 1 - e$, we get

$$\{1, v_1/1, w_1\}_f(t) = \frac{[2-(1-e)\,t](t+1-e)}{[2(1-e)+(1+2e-e^2)\,t](t+1)}$$

$$[1/1]_f(t) = (2-t)/(2+t).$$

t	1-Log$(1+t)$	$\{1, v_1/1, w_1\}_f(t)$	$[1/1]_f(t)$
−0.9	3.3025850	4.6039730	2.6363636
−0.5	1.6931471	1.7097808	1.6666666
−0.3	1.3566749	1.3584556	1.3529411
−0.1	1.1053605	1.1053971	1.1052631
0.1	0.9046898	0.9046680	0.9047619
0.3	0.7376357	0.7372726	0.7391304
0.8	0.4122133	0.4100425	0.4285714
1.5	0.0837092	0.0821464	0.1428571
1.7	0.0067482	0.0065906	0.0810810
2.5	−0.2527629	−0.2416952	−0.1111111
4	−0.6094379	−0.5587768	−0.3333333

4. Error

From the theory of Padé approximation we know that

$$h(t) - [m/n]_h(t) = \frac{h^{m+n+1}}{\tilde{q}_n(t)}\, h^{(m-n+1)}\left(\frac{x^n q_n(x)}{1-xt}\right).$$

Multiplying both sides by $\tilde{v}_k(t)/\tilde{w}_r(t)$ and using the above relation between the functionals $h^{(m-n+1)}$ and $g^{(m-n-r+1)}$, we obtain

$$f(t)-\{m, v_k/n, w_r\}_f(t)=\frac{t^{m+n+1}\tilde{v}_k(t)}{\tilde{q}_n(t)\,\tilde{w}_r(t)}\,g^{(m-n-r+1)}\left(\frac{x^n w_r(x)\,q_n(x)}{1-xt}\right).$$

In a similar way from the other expressions of the error [1] we have

$$f(t)-\{m, v_k/n, w_r\}_f(t)=\frac{t^{m+n+1}\tilde{v}_k(t)}{\tilde{q}_n(t)\,\tilde{w}_r(t)}\,g^{(m-r+1)}\left(\frac{w_r(x)q_n(x)}{1-xt}\right),$$

$$f(t)-\{m, v_k/n, w_r\}_f(t)=\frac{t^{m+n+1}\tilde{v}_k(t)}{[\tilde{q}_n(t)]^2\,\tilde{w}_r(t)}\,g^{(m-n-r+1)}\left(\frac{w_r(x)\,q_n^2(x)}{1-xt}\right).$$

5. Algebraic Properties

Since the partial Padé approximants of f are related to the Padé approximants of h, it is natural that they share some of their algebraic properties.

Let e be the reciprocal series of f formally defined by

$$f(t)e(t)=1.$$

e is assumed to exist that is c_0 is assumed to be different from zero. We obviously have

$$\{m, v_k/n, w_r\}_f(t)=1/\{n, w_r/m, v_k\}_e(t).$$

If we set $F(t)=t^s f(t)$, we know that

$$t^s\tilde{p}_m(t)/\tilde{q}_n(t)=[m+s/n]_{t^s h}(t).$$

Thus

$$\{m+s, v_k/n, w_r\}_F(t)=t^s\{m, v_k/n, w_r\}_f(t).$$

The other properties of Padé approximants dealing with value transformations do not generalize to partial Padé approximants except for the obvious one:

$$\{m, v_k/n, w_r\}_F(t)=a\{m, v_k/n, w_r\}_f(t), \qquad \text{for} \quad F(t)=af(t).$$

Let us now consider argument transformations.

We set $F(t)=f(at)$, $a\neq 0$, $\tilde{V}_k(t)=\tilde{v}_k(at)$, and $\tilde{W}_r(t)=\tilde{w}_r(at)$. We have

$$\{m, V_k/n, W_r\}_F(t)=\{m, v_k/n, w_r\}_f(at)$$

since if $H(t)=h(at)$, we know that $[m/n]_H(t)=[m/n]_h(at)$.

Now let $F(t)=f(t^s)$, $s>0$, $\tilde{V}_k(t)=\tilde{v}_k(t^s)$, and $\tilde{W}_r(t)=\tilde{w}_r(t^s)$. We have $\forall i,j$ such that $i+j\leqslant s-1$

$$\{ms+i, V_k/ns+j, W_r\}_F(t)=\{m, v_k/n, w_r\}_f(t^s).$$

Let

$$F(t)=f\left(\frac{at}{1+bt}\right), \qquad \tilde{V}_k(t)=(1+bt)^k\,\tilde{v}_k\left(\frac{at}{1+vt}\right)$$

and

$$\tilde{W}_k(t)=(1+bt)^k\,\tilde{w}_k\,\left(\frac{at}{1+bt}\right)$$

with $a\neq 0$. We have

$$\{n, V_k/n, W_k\}_F(t)=\{n, v_k/n, w_k\}_f\left(\frac{at}{1+bt}\right).$$

This property can be generalized with $\tilde{V}_k(t)=(1+bt)^k\tilde{v}_k(at/(1+bt))$ and $\tilde{W}_r(t)=(1+bt)^r\,\tilde{w}_r(at/(1+bt))$. We have, if $m+k=n+r$,

$$\{m, V_k/n, W_r\}_F(t)=\{m, v_k/n, w_r\}_f\left(\frac{at}{1+bt}\right).$$

6. Connection with Interpolation

Before looking at partial Padé approximants we shall first show how the cases of Padé-type and Padé approximants can be related to polynomial interpolation and how their numerators can be directly expressed from the functional c. This approach is different from that given in [1] since it starts from the error formula.

For Padé-type approximants we have

$$(p/q)_f(t)=c\left(\frac{1}{1-xt}\right)-\frac{t^{p+1}}{\tilde{v}_q(t)}\,c\left(x^{p-q+1}\,\frac{v_q(x)}{1-xt}\right)=\tilde{w}_p(t)/\tilde{v}_q(t)$$

$$=\frac{1}{\tilde{v}_q(t)}\,c\left(\frac{\tilde{v}_q(t)-t^{p+1}x^{p-q+1}\,v_q(x)}{1-xt}\right).$$

Thus

$$\tilde{w}_p(t) = c\left(\frac{t^q v_q(t^{-1}) - t^{p+1}x^{p-q+1} v_q(x)}{1-xt}\right)$$

$$w_p(t) = t^p \tilde{w}_p(t^{-1}) = t^p\, c\left(\frac{t^{-q} v_q(t) - t^{-p-1}\, x^{p-q+1}\, v_q(x)}{1-xt^{-1}}\right),$$

and we finally obtain

$$w_p(t) = c\left(\frac{x^{p-q+1} v_q(x) - t^{p-q+1}\, v_q(t)}{x-t}\right)$$

which generalizes a known formula when $p = q-1$.

For Padé approximants we have

$$[p/q]_f(t) = c\left(\frac{1}{1-xt}\right) - \frac{t^{p+q+1}}{\tilde{v}_q(t)}\, c\left(x^{p-q+1}\frac{x^q v_q(x)}{1-xt}\right) = \tilde{w}_p(t)/\tilde{v}_q(t)$$

$$= \frac{1}{\tilde{v}_q(t)}\, c\left(\frac{\tilde{v}_q(t) - t^{p+q+1}x^{p+1}\, v_q(x)}{1-xt}\right).$$

Thus

$$\tilde{w}_p(t) = c\left(\frac{t^q v_q(t^{-1}) - t^{p+q+1}x^{p+1}\, v_q(x)}{1-xt}\right)$$

$$w_p(t) = t^p \tilde{w}_p(t^{-1}) = t^p\, c\left(\frac{t^{-q} v_q(t) - t^{-p-q-1}\, x^{p+1}\, v_q(x)}{1-xt^{-1}}\right),$$

and we finally obtain

$$w_p(t) = t^{-q} c\left(\frac{x^{p+1}\, v_q(x) - t^{p+1}\, v_q(t)}{x-t}\right).$$

Let us now show that, due to the orthogonality property of v_q in the case of Padé approximants, both formulas for w_p are the same.

For Padé-type approximants we have

$$w_p(t) = c\Bigg((t^{p-q} v_q(t) - t^{-1}x^{p-q+1}\, v_q(x))$$
$$\times\left(1 + xt^{-1} + \cdots + x^{q-1}t^{-q+1} + \frac{x^q t^{-q}}{1-xt^{-1}}\right)\Bigg).$$

But, for Padé approximants, we have

$$c(x^{p-q+1}x^i v_q(x))=0, \qquad i=0, ..., q-1$$

and thus

$$w_p(t)=c\left(\frac{t^{p-q}v_q(t)}{1-xt^{-1}}-t^{-1}\,x^{p-q+1}\,\frac{x^q t^{-q}}{1-xt^{-1}}\,v_q(x)\right)$$

$$=t^{-q}c\left(\frac{t^p v_q(t)-t^{-1}\,x^{p+1}\,v_q(x)}{1-xt^{-1}}\right),$$

and finally we get

$$w_p(t)=t^{-q}\,c\,\left(\frac{x^{p+1}\,v_q(x)-t^{p+1}\,v_q(t)}{x-t}\right),$$

which is the formula already obtained in the Padé case. $(x^{p-q+1}v_q(x)-t^{p-q+1}v_q(t))/(x-t)$ is a polynomial of degree p in t and p in x. Thus the functional c can be applied to it and w_p is a polynomial of degree p in t as expected.

In the same way $(x^{p+1}v_q(x)-t^{p+1}\,v_q(t))/(x-t)$ is a polynomial of degree $p+q$ in t and $p+q$ in x. Thus the functional c can be applied to it and we obtain a polynomial of degree $p+q$ in t in which, due to the orthogonality property of v_q, there are no terms of degree strictly less than q. Therefore, when multiplied by t^{-q}, we get a polynomial of degree p in t as expected.

Let us now study the connection with polynomial interpolation. Starting again from the error formula we have

$$(p/q)_f(t)=c(R(x))$$

with

$$R(x)=\frac{1}{1-xt}\left(1-t^{p+1}x^{p-q+1}\,\frac{v_q(x)}{\tilde{v}_q(t)}\right).$$

Thus R is a polynomial of degree p in x. When $p=q-1$ it is the Hermite interpolation polynomial of $(1-xt)^{-1}$ at the zeros of v_q.

The same is true for Padé approximants but, thanks to the orthogonality property of v_q, we have

$$c(R(x))=c(\bar{R}(x)),$$

where

$$\bar{R}(x)=\frac{1}{1-xt}\left(1-t^{p+q+1}\,x^{p+1}\frac{v_q(x)}{\tilde{v}_q(t)}\right).$$

We shall now look at partial Padé approximants. In Section 3 the functionals $h^{(r+s)}$ and $g^{(s)}$ have been related. Let us now connect the functionals $c^{(k+s)}$ and $g^{(s)}$. From the relations between the c_i's and the g_i's it is easy to see that

$$c^{(k+s)}(x^i)=g^{(s)}(x^i v_k(x)).$$

Let $u_k(x)=x^{-k}(u_0+u_1\,x^{-1}+u_2\,x^{-2}+\cdots)$ be formally defined by

$$v_k(x)\,u_k(x)=1.$$

That is

$$x^{-k}(v_k+\cdots+v_0x^k)(u_0+u_1\,x^{-1}+\cdots)=1$$

and thus we have

$$u_0\,v_0=1$$

$$u_0v_1+u_1v_0=0$$

$$\cdots\cdots\cdots\cdots$$

$$u_0v_k+u_1\,v_{k-1}+\cdots+u_kv_0=0$$

$$u_1\,v_k+u_2\,v_{k-1}+\cdots+u_{k+1}v_0=0$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

Thus since v_k has the exact degree k, the series u_k exists. We have, from the preceding relation,

$$c^{(k+s)}(x^i u_k(x))=g^{(s)}(x^i v_k(x)\,u_k(x))$$

and thus

$$g^{(s)}(x^i)=c^{(k+s)}(x^i u_k(x)).$$

From the error formula of $\{m, v_k/n, w_r\}$ we have

$$\{m, v_k/n, w_r\}_f(t)=c\left(\frac{1}{1-xt}\right)-\frac{t^{m+n+1}\tilde{v}_k(t)}{\tilde{q}_n(t)\,\tilde{w}_r(t)}\,g^{(m-r+1)}\left(\frac{w_r(x)\,q_n(x)}{1-xt}\right)$$

$$=c\left(\frac{1}{1-xt}\right)-\frac{t^{m+n+1}\tilde{v}_k(t)}{\tilde{q}_n(t)\,\tilde{w}_r(t)}\,c^{(m+k-r+1)}\left(\frac{w_r(x)\,q_n(x)\,u_k(x)}{1-xt}\right).$$

Thus $\{m, v_k/n, w_r\}_f(t) = c(\bar{R}(x))$ with

$$\bar{R}(x) = \frac{1}{1-xt}\left(1 - \frac{t^{m+n+1}\tilde{v}_k(t)}{\tilde{q}_n(t)\,\tilde{w}_r(t)}\,x^{m+k-r+1}\,w_r(x)\,q_n(x)\,u_k(x)\right).$$

We also have

$$\{m, v_k/n, w_r\}_f(t) = \frac{1}{\tilde{q}_n(t)\,\tilde{w}_r(t)} \times c\left(\frac{\tilde{q}_n(t)\,\tilde{w}_r(t) - t^{m+n+1}\,\tilde{v}_k(t)\,x^{m+k-r+1}\,w_r(x)\,q_n(x)u_k(x)}{1-xt}\right)$$
$$= \tilde{p}_m(t)\,\tilde{v}_k(t)/\tilde{q}_n(t)\,\tilde{w}_r(t).$$

Thus we obtain the expression of $\tilde{p}_m(t)\,\tilde{v}_k(t)$. Since

$$p_m(t)\,v_k(t) = t^{m+k}\,p_m(t^{-1})\,v_k(t^{-1}),$$

we have

$$p_m(t)\,v_k(t)$$
$$= t^{m+k}\,c\left(\frac{t^{-n-r}q_n(t)\,w_r(t) - t^{-m-n-1}t^{-k}v_k(t)\,x^{m+k+r+1}\,w_r(x)\,q_n(x)\,u_k(x)}{1-xt^{-1}}\right)$$
$$= t^{-n}\,c\left(\frac{t^{m+k-r}q_n(t)\,w_r(t) - t^{-1}v_k(t)\,x^{m+k-r+1}\,w_r(x)\,q_n(x)\,u_k(x)}{1-xt^{-1}}\right).$$

We finally obtain

$$p_m(t) = t^{-n}\,c\left(\frac{x^{m+k-r+1}w_r(x)\,q_n(x)\,u_k(x) - t^{m+k-r+1}w_r(t)\,q_n(t)\,u_k(t)}{x-t}\right).$$

Since $c(x^i u_k(x)) = g^{(-k)}(x^i)$ thus p_m is a polynomial of degree m in t. If $k = r = 0$, then $u_k(x) = u_0$ and this formula reduces to that of the classical Padé case.

The orthogonality property of q_n can be written as

$$c(x^i x^{m+k-r-n+1}\,w_r(x)\,q_n(x)\,u_k(x)) = 0, \qquad i = 0, ..., n-1.$$

Due to this property we have

$$\{m, v_k/n, w_r\}_f(t) = c(\bar{R}(x)) = c(R(x)),$$

where

$$R(x)=\frac{1}{1-xt}\left(1-\frac{t^{m+1}\tilde{v}_k(t)}{\tilde{q}_n(t)\,\tilde{w}_r(t)}\,x^{m+k-r-n+1}w_r(x)\,q_n(x)\,u_k(x)\right).$$

If $m+k=n+r-1$ we have

$$R(x)=\frac{1}{1-xt}\left(1-\frac{t^{m+1}\,\tilde{v}_k(t)}{\tilde{q}_n(t)\,\tilde{w}_r(t)}\,w_r(x)\,q_n(x)\,u_k(x)\right)$$

which shows that R interpolates $(1-xt)^{-1}$ in the Hermite sense at the zeros of $q_n(x)\,w_r(x)$ since, from the relation $u_k(x)\,v_k(x)=1$, it can be seen that $u_k(x)$ has no zero because the polynomial v_k is always finite.

From the expression of R we obtain a formula similar to the above Padé-type case,

$$p_m(t)=c\left(\frac{x^{m+k-r-n+1}w_r(x)\,q_n(x)\,u_k(x)-t^{m+k-r-n+1}\,w_r(t)\,q_n(t)\,u_k(t)}{x-t}\right),$$

which is equivalent to the preceding one by the orthogonality property of q_n.

Remark. We have

$$h^{(s)}(x^i)=g^{(s-r)}(x^i w_r(x))=g^{(s)}(x^i x^{-r}\,w_r(x))=g^{(s)}(x^i\tilde{w}_r(x^{-1})).$$

Similarly $c^{(s)}(x^i)=g^{(s)}(x^i\tilde{v}_k(x^{-1}))$.

7. Two Particular Cases

Let us now choose $\tilde{v}_k(t)/\tilde{w}_r(t)=[k/r]_f(t)$. We have

$$f(t)\,\tilde{w}_r(t)-\tilde{v}_k(t)=t^{k+r+1}c^{(k-r+1)}(x^r w_r(x)/(1-xt)).$$

Dividing both sides by $\tilde{v}_k$, we get

$$h(t)=1+\frac{t^{k+r+1}}{\tilde{v}_k(t)}\,c^{(k+1)}\left(\frac{w_r(x)}{1-xt}\right).$$

Due to the block structure of the Padé table we have

$$[m/n]_h(t)=1, \qquad \text{for} \quad m, n=0, ..., k+r$$

and thus, in that case

$$\{m, v_k/n, w_r\}_f(t)=[k/r]_f(t).$$

If $m > k + r$ or $n > k + r$ we have

$$\{m, v_k/n, w_r\}_f(t) = [k/r]_f(t)[m/n]_h(t).$$

Let us give an example. We again consider the series

$$f(t) = 1 - \mathrm{Log}(1 + t).$$

We take $\tilde{v}_0(t)/\tilde{w}_1(t) = [0/1]_f(t) = 1/(1 + t)$. We have $h_0 = 1$, $h_1 = 0$, $h_2 = -\frac{1}{2}$, $h_3 = \frac{1}{6}$, and

$$[2/1]_h(t) = (6 + 2t - 3t^2)/(6 + 2t).$$

Thus

$$\{2, v_0/1, w_1\}_f(t) = (6 + 2t - 3t^2)/(6 + 8t + 2t^2).$$

t	$\{2, v_0/1, w_1\}_f(t)$	$\{3, v_1/1, w_1\}_f(t)$	$\{1, v_1/3, w_1\}_f(t)$
−0.9	4.2142857	2.9275619	2.9637199
−0.5	1.7000000	1.6898148	1.6901408
−0.3	1.3571428	1.3565224	1.3565319
−0.1	1.1053639	1.1053601	1.1053601
0.1	0.9046920	0.9046900	0.9046901
0.3	0.7377622	0.7376843	0.7376871
0.8	0.4152046	0.4155102	0.4158964
1.5	0.1000000	0.1198979	0.1230769
1.7	0.0287628	0.0631373	0.0663888
2.5	−0.2012987	−0.0468106	−0.0703812
4	−0.4857142	0.2592592	−0.1200000

$$\{3, v_1/1, w_1\}_f(t) = \frac{2-t}{2+t}\,\frac{12+6t-t^3}{12+6t}$$

$$\{1, v_1/3, w_1\}_f(t) = \frac{2-t}{2+t}\,\frac{12+6t}{12+6t+t^3}.$$

For the last two columns we have $h_0 = 1$, $h_1 = 0$, $h_2 = 0$, $h_3 = 1/12$, $h_4 = 1/24$, and $[1/1]_f(t) = (2 - t)/(2 + t)$.

Similarly we can choose

$$\tilde{v}_k(t)/\tilde{w}_r(t) = (k/r)_f(t).$$

We have

$$f(t)\,\tilde{w}_r(t) - v_k(t) = t^{k+1} c^{(k-r+1)}\left(\frac{w_r(x)}{1 - xt}\right).$$

Dividing both sides by $\tilde{v}_k$, we get

$$h(t)=1+\frac{t^{k+1}}{\tilde{v}_k(t)}c^{(k-r+1}\left(\frac{w_r(x)}{1-xt}\right).$$

Due to the block structure of the Padé table we have

$$[m/n]_h(t)=1, \qquad \text{for} \quad m, n=0,..., k$$

and thus, in that case

$$\{m, v_k/n, w_r\}_f(t)=(k/r)_f(t).$$

If $m>k$ or $n>k$ we have

$$\{m, v_k/n, w_r\}_f(t)=(k/r)_f(t)[m/n]_h(t).$$

Let us again consider the series $f(t)=1\text{-Log}(1+t)$ with

$$\tilde{v}_1(t)/\tilde{w}_2(t)=(1/2)_f(t)=(6+t)/(6+7t+2t^2).$$

We obtain $h_0=1$, $h_1=0$, $h_2=-\frac{1}{3}$, $h_3=-\frac{1}{36}$, and

$$\{2, v_1/1, w_2\}_f(t)=\frac{(6+t)(12-t-4t^2)}{(6+7t+2t^2)(12-t)}.$$

t	$\{2, v_1/1, w_2\}_f(t)$
−0.9	2.8932346
−0.5	1.6866666
−0.3	1.3561692
−0.1	1.1053563
0.1	0.9046868
0.3	0.7374581
0.8	0.4072759
1.5	0.0510204
1.7	−0.0397779
2.5	−0.3852339
4	−1.0606060

In both cases the computation of $\{m, v_k/n, w_r\}$ needs the knowledge of $c_0, ..., c_{m+n}$. Thus such partial Padé approximants have to be compared with $[p/q]_f$ where $p+q=m+n$ since they achieve the same order of approximation.

We have

$$[2/1]_f(t) = (6 - 2t - t^2)/(6 + 4t)$$
$$[3/1]_f(t) = (24 - 6t - 6t^2 + t^3)/(24 + 18t)$$
$$[2/2]_f(t) = (6 - 2t^2)(6 + 6t + t^2).$$

t	$[2/1]_f(t)$	$[3/1]_f(t)$	$[2/2]_f(t)$
−0.9	2.9125000	3.0526923	3.1063829
−0.5	1.6875000	1.6916666	1.6923076
−0.3	1.3562500	2.3566129	1.3566433
−0.1	1.1053571	1.1053603	1.1053604
0.1	0.9046875	0.9046899	0.9046898
0.3	0.7375000	0.7376530	0.7376425
0.8	0.4086956	0.4133333	0.4125874
1.5	0.0625000	0.0955882	0.0869565
1.7	−0.0226562	0.0251465	0.0115243
2.5	−0.3281250	−0.1865942	−0.2385321
4	−0.8181818	−0.3333333	−0.5652173

In both cases the computation of the coefficients of h simplifies. We no longer need the computation of the g_i's. From the expression of h, we have

$$(h_{k+1} + h_{k+2}t + \cdots)(v_0 + \cdots + v_k t^k) = \sum_{i=0}^{\infty} c^{(k-r+1)}(x^i w_r(x))\, t^i.$$

Identifying the coefficients we obtain

$$h_0 = 1, \qquad h_1 = \cdots = h_k = 0$$
$$v_0 h_{k+1} = c^{(k-r+1)}(w_r(x))$$
$$v_0 h_{k+2} + v_1 h_{k+1} = c^{(k-r+1)}(x w_r(x))$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$v_0 h_{2k+1} + v_1 h_{2k} + \cdots + v_k h_{k+1} = c^{(k-r+1)}(x^k w_r(x))$$
$$v_0 h_{2k+2} + v_1 h_{2k+1} + \cdots + v_k h_{k+2} = c^{(k-r+1)}(x^{k+1} w_r(x))$$
$$v_0 h_{2k+3} + v_1 h_{2k+2} + \cdots + v_k h_{k+3} = c^{(k-r+1)}(x^{k+2} w_r(x))$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

with

$$c^{(k-r+1)}(x^i w_r(x)) = w_0 c_{k+i+1} + w_1 c_{k+i} + \cdots + w_r c_{k-r+i+1}.$$

If $\tilde{v}_k(t)/\tilde{w}_r(t) = [k/r]_f(t)$, due to the orthogonality property of w_r, we have $c^{(k-r+1)}(x^i w_r(x)) = 0$ for $i = 0, ..., r-1$ and thus $h_{k+1} = \cdots = h_{k+r} = 0$. We have

$$v_0 h_{k+r+1} = c^{(k-r+1)}(x^r w_r(x)) = c^{(k+1)}(w_r(x))$$

$$v_0 h_{k+r+2} + v_1 h_{k+r+1} = c^{(k+1(}(x w_r(x))$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$v_0 h_{2k+r+1} + v_1 h_{2k+r} + \cdots + v_k h_{k+r+1} = c^{(k+1)}(x^k w_r(x))$$

$$v_0 h_{2k+r+2} + v_1 h_{2k+r+1} + \cdots + v_k h_{k+r+2} = c^{(k+1)}(x^{k+1} w_r(x))$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

with

$$c^{(k+1)}(x^i w_r(x)) = w_0 c_{k+r+i+1} + w_1 c_{k+r+i} + \cdots + w_r c_{k+i+1}.$$

Of course, $\tilde{v}_k(t)/\tilde{w}_r(t)$ can also be taken as the Padé-type approximant of higher order s, $k \leqslant s \leqslant k+r$. In that case similar expressions and results hold.

Let us now give an application of the above particular case of partial Padé approximants to the estimation of the error of $[k/r]_f(t)$.

From the general principles stated in [2] we know that a good estimate of the error $[k/r]_f(t) - f(t)$ is given by $[k/r]_f(t) - [k/r]_f [m/n]_h(t)$ when m and/or n is strictly greater than $k+r$.

To be more specific we know that

$$[k/r]_f(t) - \{m, v_k/n, w_r\}_f(t) = [k/r]_f(t) - f(t) + O(t^{m+n+1}),$$

$$\frac{[k/r]_f(t) - \{m, v_k/n, w_r\}_f(t)}{[k/r]_f(t) - f(t)} = 1 + O(t^{m+n-k-r}).$$

Let us compare this estimation with Kronrod's procedure. We shall take $k = r-1$ and $m+n = 3r+1$ with m and/or $n > 2r-1$. Thus the simplest case corresponds to the choice $n = 0$ and $m = 3r+1$ and we have

$$\{3r+1, v_{r-1}/0, w_r\}_f(t) = [r-1/r]_f(t)[3r+1/0]_h(t)$$

$$\{3r+1, v_{r-1}/0, w_r\}_f(t) - [r-1/r]_f(t) = [r-1/r]_f(t)(h_{3r+1}(t) - 1),$$

with $h_{3r+1}(t) - 1 = h_{2r} t^{2r} + \cdots + h_{3r+1} t^{3r+1}$, where the h_i's are given by the relations

$$v_0 h_{2r} = c(x^r w_r(x))$$

$$v_0 h_{2r+1} + v_1 h_{2r} = c(x^{r+1} w_r(x))$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$v_0 h_{3r-1} + v_1 h_{3r-2} + \cdots + v_{r-1} h_{2r} = c(x^{2r-1} w_r(x))$$

$$v_0 h_{3r} + v_1 h_{3r-1} + \cdots + v_{r-1} h_{2r+1} = c(x^{2r} w_r(x))$$

$$v_0 h_{3r+1} + v_1 h_{3r} + \cdots + v_{r-1} h_{2r+2} = c(x^{2r+1} w_r(x)).$$

These relations are quite similar to those giving the coefficients of the Stieltjes polynomial V_{k+1} in Kronrod's method [2].

Let us give some numerical examples. For $f(t) = e^t$ and $r = 1$ we have

$$\{4, v_0/0, w_1\} = -\frac{t^2}{1-t}\left(\frac{1}{2} + \frac{t}{3} + \frac{t^2}{8}\right).$$

For $r = 2$ we obtain

$$\{7, v_1/0, w_2\} = \frac{6+2t}{6-4t+t^2} t^4 \left(\frac{1}{72} + \frac{t}{270} + \frac{t^2}{648} + \frac{t^3}{6804}\right).$$

We have

t	$[0/1]_f(t)(h_4(t)-1)$	Kronrod's procedure	$e^t - [0/1]$
-3	-0.14062×10^1	-0.23684	-0.20021
-2	-0.44444	-0.21429	-0.19800
-1.5	-0.25312	-0.18461	-0.17687
-1.2	-0.18327	-0.15734	-0.15335
-1	-0.14583	-0.13433	-0.13212
-0.8	-0.11141	-0.10724	-0.10623
-0.5	-0.60764×10^{-1}	-0.60301×10^{-1}	-0.60136×10^{-1}
-0.3	-0.28471×10^{-1}	-0.28431×10^{-1}	-0.28412×10^{-1}
-0.1	-0.42538×10^{-2}	-0.42536×10^{-2}	-0.42535×10^{-2}
0.1	-0.59398×10^{-2}	-0.59400×10^{-2}	-0.59402×10^{-2}
0.3	-0.78589×10^{-1}	-0.78637×10^{-1}	-0.78713×10^{-1}
0.5	-0.34896	-0.349514	-0.35128
0.8	-0.27093×10^1	-0.27067×10^1	-0.27744×10^1
1.1	0.12317×10^2	0.12053×10^2	0.13004×10^2

t	$[1/2]_f(t)(h_7(t)-1)$	Kronrod's procedure	$e^t-[1/2]$
-3	0	0.51253×10^{-1}	0.49787×10^{-1}
-2	0.20406×10^{-1}	0.24455×10^{-1}	0.24224×10^{-1}
-1.5	0.12054×10^{-1}	0.12653×10^{-1}	0.12604×10^{-1}
-1.2	0.69604×10^{-2}	0.70900×10^{-2}	0.70766×10^{-2}
-1	0.42114×10^{-2}	0.42474×10^{-2}	0.42431×10^{-2}
-0.8	0.21682×10^{-2}	0.21755×10^{-2}	0.21745×10^{-3}
-0.5	0.46987×10^{-3}	0.47009×10^{-3}	0.47005×10^{-4}
-0.3	0.77476×10^{-4}	0.77481×10^{-4}	0.77480×10^{-4}
-0.1	0.12246×10^{-5}	0.12246×10^{-5}	0.12246×10^{-5}
0.1	0.15776×10^{-5}	0.15776×10^{-5}	0.15776×10^{-5}
0.3	0.16555×10^{-3}	0.16556×10^{-3}	0.16556×10^{-3}
0.5	0.16620×10^{-2}	0.16627×10^{-2}	0.16624×10^{-2}
0.8	0.16212×10^{-1}	0.16258×10^{-1}	0.16239×10^{-1}
1.1	0.85560×10^{-1}	0.86465×10^{-1}	0.86017×10^{-1}

For $f(t)=1/t\ \mathrm{Log}(1+t)$ and $r=1$ we have

$$\{4, v_0/0, w_1\}=\frac{2t^2}{2+t}\left(\frac{1}{12}-\frac{t}{12}+\frac{3t^2}{40}\right).$$

For $r=2$ we obtain

$$\{7, v_1/0, w_2\}=\frac{6+3t}{6+6t+t^2}\,t^4\left(\frac{1}{180}-\frac{t}{90}+\frac{19t^2}{1260}-\frac{11t^3}{630}\right).$$

We have

t	$[0/1]_f(t)(h_4(r)-1)$	Kronrod's procedure	$t^{-1}\mathrm{Log}(1+t)-[0/1]$
-0.9	0.32266	0.67805	0.74025
-0.7	0.13450	0.18000	0.18150
-0.5	0.47917×10^{-1}	0.52910×10^{-1}	0.52961×10^{-1}
-0.3	0.12185×10^{-1}	0.12445×10^{-1}	0.12446×10^{-1}
-0.1	0.97281×10^{-3}	0.97358×10^{-3}	0.97358×10^{-3}
0.1	0.72143×10^{-3}	0.72084×10^{-3}	0.72085×10^{-3}
0.3	0.50935×10^{-2}	0.49822×10^{-2}	0.49823×10^{-3}
0.5	0.12083×10^{-1}	0.10929×10^{-1}	0.10930×10^{-2}
0.7	0.22413×10^{-1}	0.17294×10^{-1}	0.17300×10^{-1}
0.9	0.38591×10^{-1}	0.23499×10^{-1}	0.23516×10^{-1}
1.5	0.16339	0.39318×10^{-1}	0.39432×10^{-1}
2	0.43333	0.49020×10^{-1}	0.49306×10^{-1}
3	0.18300×10^{1}	0.61224×10^{-1}	0.62098×10^{-1}
5	0.11012×10^{2}	0.70028×10^{-1}	0.72638×10^{-1}
7	0.34572×10^{2}	0.70342×10^{-1}	0.74841×10^{-1}

We see that the error estimates given by the preceding method are bad outside the convergence domain of the series while Kronrod's procedure still provides good values.

We have

t	$[1/2]_f(t)(h_7(t)-1)$	Kronrod's procedure	$t^{-1}\text{Log}(1+t)-[1/2]$
−0.9	0.62187×10^{-1}	0.21873	0.21800
−0.7	0.10922×10^{-1}	0.16935×10^{-1}	0.16904×10^{-1}
−0.5	0.14766×10^{-2}	0.16794×10^{-2}	0.16790×10^{-3}
−0.3	0.10320×10^{-3}	0.10529×10^{-3}	0.10529×10^{-6}
−0.1	0.72013×10^{-6}	0.72035×10^{-6}	0.72035×10^{-6}
0.1	0.43631×10^{-6}	0.43647×10^{-6}	0.43647×10^{-6}
0.3	0.22016×10^{-4}	0.22833×10^{-4}	0.22833×10^{-4}
0.5	0.80437×10^{-4}	0.11941×10^{-3}	0.11940×10^{-3}
0.7	-0.14958×10^{-3}	0.32290×10^{-3}	0.32287×10^{-3}
0.9	-0.23182×10^{-2}	0.64040×10^{-3}	0.64027×10^{-3}
1.5	−0.11128	0.21664×10^{-2}	0.21648×10^{-2}
2	−0.83809	0.38567×10^{-2}	0.38516×10^{-2}
3	-0.13383×10^{2}	0.75732×10^{-2}	0.75527×10^{-1}
5	-0.39925×10^{3}	0.14157×10^{-1}	0.14090×10^{-1}
7	-0.35569×10^{4}	0.18805×10^{-1}	0.18713×10^{-1}

In all these examples Kronrod's method gives better estimates. As we saw above q_n satisfies $g^{(m-n-r+1)}(x^i q_n(x)\, w_r(x))=0$ for $i=0,\dots,n-1$. If $n=r+1$ and if w_r is chosen such that

$$g^{(m-2r)}(x^i w_r(x))=0, \qquad i=0,\dots,r-1,$$

which means that w_r is the polynomial of degree r orthogonal with respect to $g^{(m-2r)}$, then

$$g^{(m-2r)}(x^i q_{r+1}(x)\, w_r(x))=0, \qquad i=0,\dots,r.$$

Thus q_{r+1} is a so-called Stieltjes polynomial and for $k=0$ and $m=2r$ we again find the approximants encountered in Kronrod's procedure since, in that case, the functionals c and $g^{(m-2r)}$ are identical.

8. Inverse Padé–Type Approximants

In Padé-type approximants (corresponding to $k=n=0$ in partial Padé approximants) the denominators (that is the poles) of the approximants are arbitrarily chosen and then the numerators are obtained by imposing that the expansion of the approximant matches the series as far as possible.

We shall now choose the numerators (that is the zeros) of the approximants arbitrarily and then the denominators will be obtained by imposing that the expansion of the approximant matches the series as far as possible. Such approximants, which correspond to $r=m=0$, will be called inverse Padé-type approximants, a name perfectly justified as we shall see now.

Let e be the reciprocal series of f, formally defined by $f(t)\,e(t)=1$, and let v_k be an arbitrary polynomial of degree k. We have by definition of the partial Padé approximants

$$\begin{aligned}\{n, w_0/0, v_k\}_e(t) &= e(t)+O(t^{n+1})\\ &= (n/k)_e(t)\end{aligned}$$

by definition and uniqueness of the Padé-type approximant $(n/k)_e$ with the generating polynomial v_k. But

$$\{0, v_k/n, w_0\}_f(t)=1/\{n, w_0/0, v_k\}_e(t)$$

and thus

$$\{0, v_k/n, w_0\}_f(t)=1/(n/k)_e(t),$$

where the Padé-type approximant $(n/k)_e$ is constructed from the generating polynomial v_k. The meaning of the given name clearly appears. Such approximants will be useful for series with known zeros as, for example, sin t or cos t.

9. Conclusion

As shown by the numerical examples given above, partial Padé approximants can be interesting if "good" choices of v_k and w_r are made. Thus one of the main open questions concerns this choice. The second important question is that of convergence. These two questions are certainly difficult ones as exemplified by our experience on Padé and Padé-type approximants.

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References

1. C. Brezinski, "Padé-Type Approximation and General Orthogonal Polynomials," ISNM, Vol. 50, Birkhäuser-Verlag, Basel, 1980.
2. C. Brezinski, Error estimate in Padé approximation, *in* "Proceedings of the Second International Symposium on Orthogonal Polynomials and Their Applications, Segovia 22–27, 1986," Lecture Notes in Mathematics, Springer-Verlag, Berlin, to appear.
3. S. P. Nørsett, Restricted Padé approximations to the exponential function, *SIAM J. Numer. Anal.* **15** (1978), 1008–1029.

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